VETRI VINAYAHA COLLEGE OF ENGINEERING AND TECHNOLOGY

DEPARTMENT OF MATHEMATICS
MA6459 NUMERICAL METHODS

BRANCH : EEE & CIVIL **SEMESTER: IV**

PART - A QUESTION AND ANSWER
UNIT-I
SOLUTIUON OF EQUATIONS AND EIGENVALUE PROBLEMS

1.**What is the sufficient condition and the order of convergence for iteration method for solving fixed point iteration $x = \phi(x)$ method.**

Solution: The sufficient condition is $|\phi'(x)| < 1$.The order of convergence for iterative method is 1.

2.**State Newton – Raphson iteration formula.**

Solution: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

3.**What is the order of convergence and convergence condition for Newton-Raphson method?**

Solution:

Order of convergence for Newton's method is 2.(Quadratic).

Condition for convergence is $|f(x).f''(x)| < |f'(x)|^2$.

4.**Locate the negative root of the equation** $x^3 - 2x + 5 = 0$.

Solution:

$Let \quad f(x) = x^3 - 2x + 5$

$f(-1) = (-1)^3 - 2(-1) + 5 = 6 = +ve$, $f(-2) = (-2)^3 - 2(-2) + 5 = 1 = +ve$

$f(-3) = (-3)^3 - 2(-3) + 5 = -16 = -ve$.

The root lies between -2 and -3.

5.**Locate the root of the equation** $x^2 = -4\sin x$.

Solution: $Let \quad f(x) = x^2 + 4\sin x.$ $f(-1) = 1 + 4\sin(-1) = -2.366 = -ve$,

$f(-2) = 4 + 4\sin(-2) = 0.3628 = +ve$. The root lies between -1 and -2.

6.**Write down the iterative formula for $\sqrt{N}$ in Newton's method and hence find $\sqrt{5}$.**

Solution: We know that $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$Let\ x=\sqrt{N} \Rightarrow x=N^{1/2} \Rightarrow x^2=N$$

$$Let\ f(x)=x^2-N, \quad f'(x)=2x$$

$$\therefore x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{x_n^2-N}{2x_n}=\frac{2x_n^2-x_n^2+N}{2x_n}=\frac{x_n^2+N}{2x_n}.$$

$$=\frac{1}{2}\left(x_n+\frac{N}{x_n}\right).$$

To find $\sqrt{5}$: $Let\ x=\sqrt{5} \Rightarrow x^2=5.\ Let\ f(x)=x^2-5$

$f(0)=0-5=-5=-ve,\ f(1)=1-5=-4=-ve,\ f(2)=4-5=-1=-ve,\ f(3)=9-5=+4=+ve.$

The root lies between 2 and 3. Take N=5, x_0=2.

$$We\ know\ that\ x_{n+1}=\frac{1}{2}\left(x_n+\frac{N}{x_n}\right).$$

$$x_1=\frac{1}{2}\left(x_0+\frac{N}{x_0}\right)=\frac{1}{2}\left(2+\frac{5}{2}\right)=2.25$$

$$x_2=2.2361\ , \qquad x_3=2.2361.$$

$$\therefore\ x_2=x_3.\ The\ value\ of\ \sqrt{5}\ is\ 2.2361.$$

7.**Write down the iterative formula for 1/N in Newtons method and hence find 1/26.**

Solution: We know that $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$

$$Let\ x=\frac{1}{N} \Rightarrow \frac{1}{x}=N. \quad Let\ f(x)=N-\frac{1}{x}, \quad f'(x)=\frac{1}{x^2}$$

$$\therefore x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

$$=x_n-\frac{N-\frac{1}{x_n}}{\frac{1}{x_n^2}}=x_n-\frac{Nx_n-1}{x_n}.x_n^2=x_n-Nx_n^2+x_n=2x_n-Nx_n^2$$

$$=x_n(2-Nx_n).$$

To find $\frac{1}{26}$: Let x_0=0.04, N=26.

$$We\ know\ that\ x_{n+1} = x_n(2 - Nx_n)$$

$$x_1 = x_0(2 - Nx_0)$$

$$= 0.0384.$$

$$x_2 = 0.0385.$$

$$x_3 = 0.0385.$$

$$\therefore x_2 = x_3. The\ value\ of\ \frac{1}{26}\ is\ 0.0385.$$

8.**Write down the iterative formula for p[th] root of N in Newtons method.**

Solution: We know that $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$Let\ x = \sqrt[p]{N} \Rightarrow x = N^{1/p} \Rightarrow x^p = N.$$

$$Let\ f(x) = x^p - N, \quad f'(x) = px^{p-1}$$

$$\therefore x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^{\ p} - N}{px_n^{\ p-1}}$$

$$= \frac{px_n^{\ p} - x_n^{\ p} + N}{px_n^{\ p-1}}$$

$$= \frac{x_n^{\ p}(p-1) + N}{px_n^{\ p-1}}.$$

9.**Write down the iterative formula for cube root of N** $\left(\sqrt[3]{N}\right)$ **in Newtons method.**

Solution: We know that $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$Let\ x = \sqrt[3]{N} \Rightarrow x = N^{1/3} \Rightarrow x^3 = N$$

$$Let\ \ f(x) = x^3 - N, \quad f'(x) = 3x^2$$

$$\therefore x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 - N}{3x_n^2} = \frac{3x_n^3 - x_n^3 + N}{3x_n^2} = \frac{2x_n^3 + N}{3x_n^2}$$

$$= \frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right).$$

10. **What are the direct methods to solve the linear simultaneous equations AX=B.**
 Solution:
 i) Gauss-Elimination Method, ii) Gauss-Jordan Method.
11. **To which forms are augmented matrices transformed in the Gauss-Elimination and Gauss-Jordan Method. (Or) Compare Gauss-Elimination and Gauss-Jordan Method.**
 Solution:

Gauss-Elimination Method	**Gauss-Jordan Method**
i) The Augmented matrix is reduced to upper triangular matrix.	i) The Augmented matrix is reduced to diagonal matrix or unit matrix.
ii) By using back substitution we get the solution.	ii)we get the solution directly.

12. **Solve the following system by Gauss-Elimination method** $2x + y = 3,\ 7x - 3y = 4.$
 Solution:
 The Augmented matrix is

$$\begin{pmatrix} 2 & 1 & 3 \\ 7 & -3 & 4 \end{pmatrix}$$

$$R_2 \to 2R_2 - 7R_1$$

$$\begin{pmatrix} 2 & 1 & 3 \\ 0 & -13 & -13 \end{pmatrix}$$

$$\therefore By\ using\ back\ substitution\ we\ get$$

$$-13y = -13 \to (1)$$

$$2x + y = 3 \quad \to (2)$$

$$From\ (1)\ \ y = 1.$$

$$(2) \Rightarrow 2x = 3 - 1 \Rightarrow x = 1.$$

$$\therefore The\ solution\ is\ x = 1, y = 1.$$

13. **By Gauss Jordan method solve** $3x+2y=4,\ 2x-3y=7.$

Solution:

The Augmented matrix is

$$\begin{pmatrix} 3 & 2 & 4 \\ 2 & -3 & 7 \end{pmatrix}$$

$$R_2 \to 3R_2 - 2R_1$$

$$\begin{pmatrix} 3 & 2 & 4 \\ 0 & -13 & 13 \end{pmatrix}$$

$$R_2 \to -\frac{R_2}{13}$$

$$\begin{pmatrix} 3 & 2 & 4 \\ 0 & 1 & -1 \end{pmatrix}$$

$$R_1 \to R_1 - 2R_2$$

$$\begin{pmatrix} 3 & 0 & 6 \\ 0 & 1 & -1 \end{pmatrix}$$

$$R_1 \to \frac{R_1}{3}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

From this we get $x=2,\ y=-1.$

$\therefore$ *The solution is* $x=2, y=-1.$

14. **What is meant by diagonally dominant?**

Solution: We say a matrix is diagonally dominant if the numerical value of the leading diagonal element in each row, is greater than or equal to the sum of the numerical values of the other elements in that row.

15. **State the sufficient condition for convergence of Gauss-Seidal iteration method for solving system of equations.**

Solution: Let the given equation be

$$a_1x+b_1y+c_1z=d_1,\ a_2x+b_2y+c_2z=d_2,\ a_3x+b_3y+c_3z=d_3$$

The sufficient condition is

$$|a_1| \ge |b_1|+|c_1|,\ \ |b_2| \ge |a_2|+|c_2|,\ \ |c_3| \ge |a_3|+|b_3|.$$

16. **What is the difference between direct and indirect method. (or) Compare Gauss-Jordan and Gauss-Seidal method.**

Solution:

Direct Method (Gauss-Jordan)	**Indirect Method (Gauss-Seidal)**
i)It gives exact value. ii)Simple take less time. iii) This method determine all the roots at	i) It gives approximate value. ii) Time consuming iii) This method determine only one

the same time.	root at a time.

17. **Find the inverse of the coefficient matrix by Gauss-Jordan elimination method.** $5x-2y=10,\ 3x+4y=12.$

Solution: The coefficient matrix is $\begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix}$.Let $A=\begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix}$

The Augmented matrix is

$$\begin{pmatrix} 5 & -2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{pmatrix}$$

$$R_2 \to 5R_2 - 3R_1$$

$$\begin{pmatrix} 5 & -2 & 1 & 0 \\ 0 & 26 & -3 & 5 \end{pmatrix}$$

$$R_1 \to 13R_1 + R_2$$

$$\begin{pmatrix} 65 & 0 & 10 & 5 \\ 0 & 26 & -3 & 5 \end{pmatrix}$$

$$R_1 \to \frac{R_1}{65},\ R_2 \to \frac{R_2}{26}$$

$$\begin{pmatrix} 1 & 0 & \frac{2}{13} & \frac{1}{13} \\ 0 & 1 & \frac{-3}{26} & \frac{5}{26} \end{pmatrix}$$

$$\therefore \text{The Inverse Matrix is} \begin{pmatrix} \frac{2}{13} & \frac{1}{13} \\ \frac{-3}{26} & \frac{5}{26} \end{pmatrix}.$$

18. **Define Eigenvalue and Eigenvector.**

Solution: Let A=[a_{ij}] be a square matrix of order n. If there exists a non-zero vector X and a scalar λ such that $AX=\lambda X$ then λ is called an eigenvalue of a and X is called an Eigenvector corresponding to the eigenvalue λ.

Power Method is used to find the Largest (Dominant) Eigenvalue and Eigenvector.

19. **Find the dominant eigenvalues and the corresponding eigenvectors by Power Method.**

Solution: $Let\ A=\begin{pmatrix} 10 & 2 & 1 \\ 2 & 10 & 1 \\ 1 & 2 & 10 \end{pmatrix}$.$Let\ X_0=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$AX_0 = \begin{pmatrix} 10 & 2 & 1 \\ 2 & 10 & 1 \\ 1 & 2 & 10 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 13 \\ 13 \\ 13 \end{pmatrix} = 13\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 13X_1$$

$$AX_1 = \begin{pmatrix} 10 & 2 & 1 \\ 2 & 10 & 1 \\ 1 & 2 & 10 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 13 \\ 13 \\ 13 \end{pmatrix} = 13\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 13X_2$$

$$\therefore \text{The largest eigenvalue is } 13. \text{ Corresponding eigenvector is } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

20. **Using Gauss-Jacobi find eigenvalues and eigenvectors of** $\begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$**.**

Solution:

$Let\ \ A = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$**.** Choose -1 in a_{12} is the largest non-diagonal element.

$$\therefore\ s_1 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad where\ \theta = \frac{-\pi}{4}. \quad [\because a_{12} = -1 = -ve]$$

$$s_1 = \begin{pmatrix} \cos\left(\frac{-\pi}{4}\right) & -\sin\left(\frac{-\pi}{4}\right) \\ \sin\left(\frac{-\pi}{4}\right) & \cos\left(\frac{-\pi}{4}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$Consider\ \ B_1 = s_1^T A s_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$$

B_1 is a Diagonal Matrix.

The Eigenvalues are 3,5.

To find the Eigenvectors:

The eigenvector is given by column's of the matrix $s_1 s_2 ... = s$.

The Eigenvectors are $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$**.**

UNIT-II INTERPOLATION AND APPROXIMATION

1.**Which methods are used to find the polynomial if the intervals are unequal and equal?**

Solution: For unequal and equal intervals we use i) Lagrangian Interpolation Formula and ii) Newton's Divided Difference Formula.

For equal intervals we use i) Nweton's Forward Difference Formula and ii) Newton's Backward Difference Formula.

2.**What do you understand by inverse interpolation?**

Solution: Inverse Interpolation is the process of finding the values of x corresponding to the value of y not present in the table.

3.**State interpolation and extrapolation.**

Solution: Interpolation is the process of finding the values inside the given range $[x_0,x_n]$.

Extrapolation is the process of finding the values outside the given range $[x_0,x_n]$.

4.**State Lagrangian Interpolation formula.**

Solution: Lagrangian Interpolation Formula to find y is

$$y(x)=\frac{(x-x_1)(x-x_2)(x-x_3).....(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3).....(x_0-x_n)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3).....(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3).....(x_1-x_n)}y_1$$

$$+.....+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3).....(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2).....(x_n-x_{n-1})}y_n$$

5.**What is the Lagrangian formula used to find y, if three sets of values (x_0,y_0), (x_1,y_1) and (x_2,y_2).**

Solution: Lagrangian Interpolation Formula to find y is

$$y(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1+\frac{(x-x_0)(x-x_1)}{(x_n-x_0)(x_n-x_1)}y_3$$

6.**Write Lagrangian inverse interpolation formula.**

Solution: Lagrangian Interpolation Formula to find x is

$$x=\frac{(y-y_1)(y-y_2)(y-y_3).....(y-y_n)}{(y_0-y_1)(y_0-y_2)(y_0-y_3).....(y_0-y_n)}x_0+\frac{(y-y_0)(y-y_2)(y-y_3).....(y-y_n)}{(y_1-y_0)(y_1-y_2)(y_1-y_3).....(y_1-y_n)}x_1$$

$$+.....+\frac{(y-y_0)(y-y_1)(y-y_2)(y-y_3).....(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)(y_n-y_2)(y_n-y_3).....(y_n-y_{n-1})}x_n$$

7.**Define Divided Difference.**

Solution: Let the function y=f(x) take the values $f(x_0), f(x_1),...f(x_n)$ corresponding to the values $x_0, x_1,...x_n$ then the

First divided difference is $f(x_0,x_1)=\dfrac{f(x_1)-f(x_0)}{x_1-x_0}$.

Second divided difference is $f(x_0,x_1,x_2)=\dfrac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0}$.

8.**State any two properties of divided difference.**

Solution: i) The operator Δ is linear.

ii) The divided difference are symmetrical in all their arguments.

iii)The n[th] divided differences of a polynomial of the n[th] degree are constant.

9.**State Newton's divided difference formula.**

Solution: Newton's divided difference Interpolation Formula is

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)$$
$$+.....+(x-x_0)(x-x_1)...(x-x_{n-1})f(x_0,x_1,x_2...x_n)$$

10. **State Newton's Forward and Backward interpolation formula.**

Solution: Newton's Forward difference Interpolation Formula is

$$y=f(x)=y_0+p\Delta y_0+\frac{p(p-1)}{2}\Delta^2 y_0+\frac{p(p-1)(p-2)}{3}\Delta^3 y_0+...A$$
$$+\frac{p(p-1)(p-2)...((p-(n-1))}{n}\Delta^n y_0 \quad where \;\; p=\frac{x-x_0}{h}$$

Newton's Backward difference Interpolation Formula is

$$y=f(x)=y_n+p\nabla y_n+\frac{p(p+1)}{2}\nabla^2 y_n+\frac{p(p+1)(p+2)}{3}\nabla^n y_n+...$$
$$+\frac{p(p+1)(p+2)...((p+(n-1))}{n}\nabla^n y_n \quad where \;\; p=\frac{x-x_n}{h}$$

11. **Derive Newton's Forward and Backward difference formula by using operator method.**

Solution:

Newton's Forward difference Formula by operator method

$$y=f_n(x)=f_n(x_0+ph)=E^p f_n(x_0)=E^p(y_0)=(1+\Delta)^p y_0 \;[\because E=(1+\Delta)]$$
$$=y_0+p\Delta y_0+\frac{p(p-1)}{2}\Delta^2 y_0+\frac{p(p-1)(p-2)}{3}\Delta^3 y_0+...$$
$$+\frac{p(p-1)(p-2)...((p-(n-1))}{n}\Delta^n y_0 \quad where \;\; p=\frac{x-x_0}{h}$$

Newton's Backward difference Formula by operator method

$$y = f_n(x) = f_n(x_n + ph) = E^p f_n(x_n) = E^p(y_n) = (1-\nabla)^{-p} y_n \quad [\because E = (1-\nabla)^{-1}]$$

$$= y_n + p\nabla y_n + \frac{p(p+1)}{2}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3}\nabla^3 y_n + \ldots$$

$$+ \frac{p(p+1)(p+2)\ldots((p+(n-1))}{n}\nabla^n y_n \qquad where \;\; p = \frac{x - x_n}{h}$$

12. **If y(x$_i$)=y$_i$, i=0,1,2,…,n. Write down the formula for the cubic spline polynomial y(x) valid in** $x_{i-1} \le x \le x_i$ **.**

Solution: We know that $M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\left[y_{i-1} - 2y_i + y_{i+1}\right]$ where i=1,2,…n-1

$$y(x) = S(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + \left(x - x_{i-1}\right)^3 M_i\right]$$

$$+\frac{1}{h}\left[(x_i - x)\left(y_{i-1} - \frac{h^2}{6}M_{i-1}\right)\right] + \frac{1}{h}\left[(x - x_{i-1})\left(y_i - \frac{h^2}{6}M_i\right)\right] \;\; where \; i = 1, 2, \ldots n$$

13. **What is cubic spline and natural cubic spline.**
Solution:A cubic spline is a polynomial which has continuous slope and curvature is called a ***cubic spline.*** A cubic spline fitted to the given data such that the end cubics are approach to zero is called a ***natural cubic spline.***
14. **State the conditions require for natural cubic spline.**
Solution: A cubic spline S(x) fits to each of the points is continuous and is continuous in slope and curvature such that $S_0''(x_0) = M_0 = 0 \;\; and \;\; S''(x_n) = M_n = 0.$ is called a natural cubic spline.

UNIT-III NUMERICAL DIFFERENTIATION AND INTEGRATION

1. **Write the formula for** $\left(\frac{dy}{dx}\right)_{x=x_0}$, $\left(\frac{d^2y}{dx^2}\right)_{x=x_0}$ *and* $\left(\frac{d^3y}{dx^3}\right)_{x=x_0}$ **using Newton's Forward difference operator.**

Solution:

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \ldots\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \ldots\right]$$

2. Write the formula for $\left(\frac{dy}{dx}\right)_{x=x_n}, \left(\frac{d^2y}{dx^2}\right)_{x=x_n}$ and $\left(\frac{d^3y}{dx^3}\right)_{x=x_n}$ using Newton's Backward difference operator.

Solution:

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \ldots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \ldots\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \ldots\right]$$

3. Write the formula for $\left(\frac{dy}{dx}\right)_{x=x_0+ph}, \left(\frac{d^2y}{dx^2}\right)_{x=x_0+ph}$ and $\left(\frac{d^3y}{dx^3}\right)_{x=x_{0+ph}}$ using Newton's Forward difference formula.

Solution:

$$\left(\frac{dy}{dx}\right)_{x=x_0+ph} = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 + \frac{2p^3-9p^2+11p-3}{12}\Delta^4 y_0 + \ldots\right]$$

$$where\ p = \frac{x-x_0}{h}$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0+ph} = \frac{1}{h^2}\left[\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{6p^2-18p+11}{12}\Delta^4 y_0 - \ldots\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_0+ph} = \frac{1}{h^3}\left[\Delta^3 y_0 + \frac{2p-3}{2}\Delta^4 y_0 + \ldots\right]$$

4. Write the formula for $\left(\frac{dy}{dx}\right)_{x=x_n+ph}, \left(\frac{d^2y}{dx^2}\right)_{x=x_n+ph}$ and $\left(\frac{d^3y}{dx^3}\right)_{x=x_n+ph}$ using Newton's Backward difference formula.

Solution:

$$\left(\frac{dy}{dx}\right)_{x=x_n+ph} = \frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2}\nabla^2 y_n + \frac{3p^2+6p+2}{6}\nabla^3 y_n + \frac{2p^3+9p^2+11p+3}{12}\nabla^4 y_n + \ldots\right]$$

where $p = \frac{x - x_n}{h}$.

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n+ph} = \frac{1}{h^2}\left[\nabla^2 y_n + (p+1)\nabla^3 y_n + \frac{6p^2+18p+11}{12}\nabla^4 y_n + \ldots\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_n+ph} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{2p+3}{2}\nabla^4 y_n + \ldots\right]$$

5. **Which methods are used to find the derivatives if the intervals are unequal.**

Solution: Lagrangian and Newton's divided difference formula.

Newton's divided difference formula is

$$y(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + \ldots$$

6. **State Trapezoidal, Simpson's 1/3 and Simpson's 3/8 rule for solving** $\int_a^b f(x)dx$.

Solution:

$$\text{Trapezoidal Rule}: \int_a^b f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\right]$$

$$\text{Simpson's 1/3 Rule}: \int_a^b f(x)dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \ldots) + 2(y_2 + y_4 + \ldots)\right]$$

$$\text{Simpson's 3/8 Rule}: \int_a^b f(x)dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + y_7 \ldots) + 2(y_3 + y_6 + \ldots)\right]$$

7. **What is the truncation error and order of error in Trapezoidal, Simpson's 1/3 and Simpson's 3/8 rule.**

Soluiton: Trapezoidal Rule: Error is $E < \frac{-(b-a)h^2}{12} y''(\varepsilon)$. Order of Error h^2.

Simpson's 1/3 and 3/8 Rule: Error is $E < \frac{-(b-a)h^4}{180} y'''(\varepsilon)$. Order of Error h^4.

8. **When Simpson's 1/3 and 3/8 rule can be applied. (or) What is the condition for Simpson's 1/3 and 3/8 rule .**

Solution: Simpson's 1/3 rule is applied only for number of intervals is even.

Simpson's 3/8 rule is applied only for number of intervals is multiple of 3.

9. **State the basic principle or approximation used in Trapezoidal rule. (or) Why trapezoidal rule is so called?**

Solution: The Trapezoidal rule is so called, because it approximates the integral by the sum of n trapezoids.

10. **What do you mean by Numerical Differentiation and integration?**

Solution:

Numerical differentiation is the process of finding the derivatives of a given function by means of a table of given values of that function.

Numerical integration is the process of evaluating a definite integral from a given set of tabulated values of the integrand f(x).

11. State Trapezoidal rule for evaluate the double integral $I = \int_a^b \int_c^d f(x)dxdy.$

Solution:

$$Trapezoidal\ Rule\ for\ 4\ Points:$$

$$I = \frac{hk}{4}\left[sum\ of\ values\ of\ f\ at\ four\ corners\right].$$

$$Extension\ Formula:$$

$$I = \frac{hk}{4}\begin{bmatrix}(sum\ of\ values\ of\ f\ at\ four\ corners) \\ +2(sum\ of\ values\ of\ f\ at\ boundaries\ except\ the\ corners) \\ +4(sum\ of\ values\ of\ f\ at\ interior\ values)\end{bmatrix}$$

12. State the Simpson's rule for evaluate the double integral. $I = \int_a^b \int_c^d f(x)dxdy.$

Solution:

$$Simpson's\ Rule\ for\ 9\ Points:$$

$$I = \frac{hk}{9}\begin{bmatrix}(sum\ of\ values\ of\ f\ at\ four\ corners) \\ +4(sum\ of\ values\ of\ f\ at\ boundaries\ except\ the\ corners) \\ +16(centre\ point)\end{bmatrix}.$$

$$Extension\ Formula:$$

$$I = \frac{hk}{9}\begin{bmatrix}(sum\ of\ values\ of\ f\ at\ four\ corners) \\ +2(sum\ of\ values\ of\ f\ at\ the\ odd\ positions\ on\ the\ boundaries\ except\ the\ corners) \\ +4(sum\ of\ values\ of\ f\ at\ the\ even\ positions\ on\ the\ boundaries\ except\ the\ corners) \\ +4(centre\ Point) \\ +8(sum\ of\ values\ of\ f\ at\ the\ even\ positions\ on\ the\ odd\ rows\ except\ boundaries) \\ +4(sum\ of\ values\ of\ f\ at\ the\ odd\ positions\ on\ the\ even\ rows\ except\ boundaries) \\ +16(sum\ of\ values\ of\ f\ at\ the\ even\ positions\ on\ the\ even\ rows\ except\ boundaries)\end{bmatrix}$$

13. Write the formula for Romberg's method.

Solution: $I = I_2 + \frac{1}{3}\left(I_2 - I_1\right)$

14. Write down the formula for Gaussian Quadrature two point and Three point formula for $\int_{-1}^{1} f(x)dx$**.**

Solution:

$$Two\ \ point\ \ formula: \int_{-1}^{1} f(x)dx = f\left(-\sqrt{\frac{1}{3}}\right) + f\left(\sqrt{\frac{1}{3}}\right).$$

$$Three\ \ point\ \ formula: \int_{-1}^{1} f(x)dx = \frac{8}{9} f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

15. From the following table find the area bounded by the curve and the x-axis from x=2 to x=7

x	2	3	4	5	6	7
F(x)	8	27	64	125	216	343

Solution:

Here No. of Intervals = 5.

By Trapezoidal rule

$$\int_{x_0}^{x_n} ydx = \frac{h}{2}\left[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)\right]$$

Area $= \frac{1}{2}\left[(8+343) + 2(27+64+125+216)\right]$

$= 607.5\, sq.units$

UNIT - IV

INITIAL VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

1. **What is meant my Single step method (Self starting method) and Multi step method (Not self starting method)?**

Solution:

Single Step Method is self starting since we can take the value which lies in the given interval [a,b] in which the root lies.

Multi step Method is not self starting since we should know any 4 values prior to the value which we needed.

2. **Mention single step methods and multi step methods.**

Solution:

Single Step Methods:

i) Taylor series Method

ii) Euler and Modified Euler's Method

iii) R-K 4th order Method.

Multi Step Methods:

i) Milne's Predictor and Corrector Method

ii) Adam's Bashforth Predictor and Corrector Method.

3. State Taylor's series method formula.

Solution:

Let us consider the equation $y'=f(x,y)$ with $y(x_0)=y_0$.

Then Taylor's series is

$$y(x) = y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \ldots$$

4. State Euler's Method formula.

Solution:

Let us consider the equation $y'=f(x,y)$ with $y(x_0)=y_0$.

Then Euler's Method is $y(x) = y_0 + hf(x_0, y_0)$

5. State Modified Euler's formula.

Solution:

Let us consider the equation $y'=f(x,y)$ with $y(x_0)=y_0$.

Then Euler's Method is

$$y(x) = y_0 + \frac{h}{2} f\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right]$$

6. What are the limitations of Euler's method.

Solution:

In Euler's method, if h is small, the method is too slow and if h is large, it gives inaccurate value.

7. Why Taylor's series method is called single step method.

Solution:

Since in this method y is approximated by a truncated series and each term of the series is function of x, from which the value of y can be obtained by direct substitution.

8. Write Runge-Kutta fourth order method for solving $y'=f(x,y), y(x_0)=y_0$.

Solution:

Let us consider the equation $y'=f(x,y)$ with $y(x_0)=y_0$.

Then R-K Fourth order Method is

$$y(x) = y(x_0+h) = y_0 + \Delta y \quad where\ \Delta y = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf\left(x_0 + h, y_0 + k_3\right)$$

9. Write Runge-Kutta fourth order method for solving simultaneous first order equations

Solution:

Let us consider the equation $y'=f_1(x,y,z)$ with $y(x_0)=y_0$ and $z'=f_2(x,y,z)$ with $z(x_0)=z_0$

Then R-K Fourth order Method is

$$y(x) = y(x_0 + h) = y_0 + \Delta y \qquad z(x) = z(x_0 + h) = z_0 + \Delta z$$

$$where\ \Delta y = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \qquad where\ \Delta z = \frac{1}{6}\left(l_1 + 2l_2 + 2l_3 + l_4\right)$$

$$k_1 = hf_1(x_0, y_0, z_0) \qquad l_1 = hf_2(x_0, y_0, z_0)$$

$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) \qquad l_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) \qquad l_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_4 = hf_1\left(x_0 + h, y_0 + k_3, z_0 + l_3\right) \qquad l_4 = hf_2\left(x_0 + h, y_0 + k_3, z_0 + l_3\right)$$

10. Write Milne's predictor and corrector formula.

Milne's predictor formula: $y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left(2y'_{n-2} - y'_{n-1} + 2y'_n\right)$

Milne's corrector formula: $y_{n+1,c} = y_{n-1} + \frac{h}{3}\left(y'_{n-1} + 4y'_n + y'_{n+1}\right)$

11. Write Adam's Bashforth predictor and corrector formula.

Adam's predictor formula: $y_{n+1,p} = y_n + \frac{h}{24}\left(55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}\right)$

Adam's corrector formula: $y_{n+1,c} = y_n + \frac{h}{24}\left(9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}\right)$

12. How many prior values are required to predict the next value in Milne's and Adam's Method. Ans: Four values

13. What is the order of truncation error in Euler's and modified Euler's method.

Solution: Order of Error in Euler's Method= h^2.Order of Error in Modified Euler's Method= h^3 . Error = $\frac{h^3}{12}.constant$.

14. What is the order of truncation error in R-K method.

Solution: Order of Error= h^5

15. What is the error term in Milne's and Adam's predictor formula.

Solution: $Error\ Term = \frac{14h}{45}\Delta^4 y_0'$

16. What is the error term in Milne's and Adam's corrector formula.

Solution: $Error\ Term = -\frac{h}{90}\Delta^4 y_0'$

17. What is the predictor-corrector method of solving a differential equation?

Solution:

Predictor-corrector methods are methods which require the values of y at $x_n, x_{n-1}, x_{n-2}, \ldots$ for computing the value of y at x_{n+1}. We first use a formula to find the value of y at x_{n+1} and this is known as **Predictor Formula.** The value of y so got is improved or corrected value by another formula is known as **Corrector Formula.**

18. Write the merits and demerits of the Taylor's method.

Merits:

i) It is easily derived for any order according to our interest.

ii) The values of y(x) for any x (x need not be a grid points) are easily obtained.

Demerits:

This method suffers from the time consumed in calculating the higher derivatives.

19. Which is better Taylor's method or R-K method?

R-K Method is better. Since

i) The use of R-K method gives quick convergence to the solutions of the differential equations than Taylor series method.

ii) In R-K method, the derivatives of higher order are not required for calculation as in Taylor's Method.

iii) R-K Methods are designed to give greater accuracy and they possess the advantage of requiring only the function values at some selected points on the sub-interval.

20. What are the properties of R-K method.

i) It is a single step method.

ii) R-K Method do not require prior calculation of the higher derivatives of y(x).

iii) It involves the computation of f(x,y) at various positions.

21. State True or False:

a. In Euler's method, if h is small, the method is too slow and if h is large, it gives inaccurate value.

b. The Modified Euler's method is based on the average of points.

Solution: i) True ii) True

22.Compare the R-K method and Predictor – Corrector method

Runge-Kutta Method:

i) R-K Methods are self starting

ii) In this method it is not possible to get truncation error.

Predictor-Corrector Method:

i) Not-self starting method. Since this method requires 4 prior values.

ii) In this method it is possible to get truncation error.

Unit-V

BOUNDRY VALUE PROBLEMS IN ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

1. State the Finite difference approximation for y' and y'' with error terms.

$$Let\ \ y_i = y(x_i)\ \ and\ \ x_{i+1} = x_i + h,\ i = 0,1,...,n.$$

$$Then\ y_i' = \frac{y_{i+1} - y_{i-1}}{2h}, \quad Error = o(h^2)$$

$$y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}, \ \ Error = o(h^2)$$

2. Solve by Finite difference method $y'' = y, \ \ y(0) = -1, \ y(2) = 15$ **taking h=1**

Sol: Given h=1. $y(0) = -1, \ y(2) = 15$

x	0	1	2
y	-1	?	15

Given $y'' - y = 0$ ....(1)

$$we\ \ know\ that\ \ y'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} \text{...........}(2)$$

Substituting (2) in (1), we get $\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} - y_i = 0$

$$y_{i-1} - 2y_i + y_{i+1} - h^2 y_i = 0$$

$$For\ \ i = 1,\ \ y_0 + y_2 - (2 + h^2)y_1 = 0 \qquad \left[\because\ y_0 = -1, y_2 = 15,\ h = 1\right]$$

$$-1 + 15 - (2+1)y_1 = 0 \ \ \Rightarrow -3y_1 = -14 \ \ \Rightarrow\ y_1 = 4.6667.$$

3. Write down the Finite difference scheme for the solution of the boundary value problem $y'' + y = 0, \ \ y(0) = 0, \ y(\frac{\pi}{2}) = 1$.

Sol: Given $y'' - y = 0$ ....(1)

$$we\ \ know\ that\ \ y'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} \text{...........}(2)$$

Substituting (2) in (1), we get $\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + y_i = 0 \Rightarrow y_{i-1} + y_{i+1} + (h^2 - 2)y_i = 0$

4. Obtain the Finite difference scheme for the ordinary differential equation $2\frac{d^2y}{dx^2} + y = 5$.

Sol: $2y''(x) + y(x) = 5$ ....(1)

$$we\ \ know\ that\ \ y'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} \text{...........}(2)$$

Substituting (2) in (1), we get

$$2\left(\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}\right)+y_i=5 \Rightarrow 2y_{i-1}-4y_i+2y_{i+1}+h^2y_i=5h^2$$

$$2y_{i-1}+2y_{i+1}+(h^2-4)y_i=5h^2$$

5. Define a difference quotient.

Sol: A difference quotient, is the quotient obtained by dividing the difference between two values of a function, by the difference between the two corresponding values of the independent variable.

6. State the conditions for the equation. $Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=G$ **where A,B,C,D,E,F,G are function of** x *and* y **to be (i)Elliptic (ii)Parabolic (iii)Hyperbolic**.

Solution: The given equation is said to be

(i) Elliptic if $B^2-4AC<0$

(ii) Parabolic if $B^2-4AC=0$

(iii) Hyperbolic if $B^2-4AC>0$

7. State the conditions for the equation $Au_{xx}+2Bu_{xy}+Cu_{yy}=f(u_x,u_y,x,y)$ **to be (a) elliptic (b) parabolic (c) hyperbolic when A,B,C are functions of x and y.**

Solution: The equation is elliptic if $(2B)^2-4AC<0$ i.e., $B^2-AC<0$.

It is parabolic if $B^2-AC=0$ and hyperbolic if $B^2-AC>0$.

8. Classify the equation:

$i)\ u_{xx}+2u_{xy}+u_{yy}=0.$ $ii)\ xf_{xx}+yf_{yy}=0\ \ x>0;\ y>0.$

$iii)\ f_x-f_{yy}=0.$ $iv)\ u_{xx}+4u_{yy}+3u_{xy}+4u_y+3u_x=0.$

Soln :

$i)\ u_{xx}+2u_{xy}+u_{yy}=0.$

Here $A=1, B=2, C=1$

$\therefore B^2-4AC=4-4(1)(1)=0$

It is an **Parabolic equation.**

$iii)\ f_x-f_{yy}=0..$

Here $A=0, B=0, C=-1$

$\therefore B^2-4AC=0-4(0)=0$

It is an **Parabolic equation.**

$ii)\ xf_{xx}+yf_{yy}=0\ \ x>0;\ y>0.$

Here $A=x, B=0, C=y$

$\therefore B^2-4AC=0-4(x)(y)$

$=-4xy<0\quad [\because x>0, y>0]$

It is an **Elliptic equation.**

$iv)\ u_{xx}+4u_{yy}+3u_{xy}+4u_y+3u_x=0.$

Here $A=1, B=3, C=4$

$\therefore B^2-4AC=9-4(1)(4)=-7<0$

It is an **Elliptic equation.**

9. Classify the equation $x^2f_{xx}+(1-y^2)f_{yy}=0. -\infty<x<\infty, -1<y<1..$

Soln :

$x^2 f_{xx} + (1-y^2) f_{yy} = 0$

Here $A = x^2, B = 0, C = 1 - y^2$

$\therefore B^2 - 4AC = 0 - 4\left(x^2\right)\left(1-y^2\right)$

$B^2 - 4AC = -4x^2\left(1-y^2\right).$

$Case\ (i)\ : If\ \ x = 0,\ B^2 - 4AC = 0.\ \ \therefore It\ is\ Parabolic.$

$Case\ (ii) : If\ \ y = 1\ or\ y = -1,\ B^2 - 4AC = 0.\ \therefore It\ is\ Parabolic.$

$Case\ (iii) : If\ \ -\infty < x < \infty,\ -1 < y < 1,\ B^2 - 4AC = -ve.\ \therefore It\ is\ Elliptic$

10. Give an example of a elliptic equation ,parabolic equation and hyperbolic equation.

Solution: Example for

i) **Elliptic:** The two dimensional Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

The two dimensional Poisson equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$

ii) **Parabolic** :The one dimensional heat equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$

iii) **Hyperbolic:** The one dimensional wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial^2 u}{\partial t^2}$

11. What is the classification of one dimensional heat flow equation, one dimensional wave equation, two dimensional Laplace and Poisson equation?

Solution:

i) One dimensional heat flow equation is $\frac{\partial^2 u}{\partial x^2} = a\frac{\partial u}{\partial t}$ $\quad ie) u_{xx} - au_t = 0$

Here A =1 , B = 0 , C = 0 $\therefore\ B^2 - 4AC = 0 - 4(1)(0) = 0$

Hence the one dimensional heat flow equation is Parabolic.

ii) One dimensional wave equation is $\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial^2 u}{\partial t^2}$ $\quad ie) \alpha^2 u_{xx} - u_{tt} = 0$

Here A = α^2 , B = 0 , C = -1 $\therefore\ B^2 - 4AC = 0 - 4(\alpha^2)(-1) = 4(\alpha^2) > 0.$

Hence the one dimensional heat flow equation is Hyperbolic.

iii)Two dimensional Laplace and Poisson equation:

Laplace equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, **Poisson equation is** $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$

In both equations A=1, B=0, C=1, $\therefore\ B^2 - 4AC = 0 - 4(1)(1) = -4 < 0.$

Hence the two dimensional Laplace and Poisson equation are Elliptic

12. Name the two methods that you can use to solve One dimensional heat equation (Parabolic equation).

Soln: i) Bender Schmidt method ii) Crank- Nicholson method**.**

13. State Bender Schmidt's explicit formula for solving heat flow $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$. *(0r)* Write Explicit formula to solve One dimensional heat equation (Parabolic equation) for $\lambda = \frac{1}{2}$.*(0r)* Write Bender Schmidt's recurrence equation for $\lambda = \frac{1}{2}$.

Solution:

Explicit Formula

$$u_{i,\, j+1} = \lambda u_{i+1,\, j} + (1-2\lambda) u_{i,\, j} + \lambda\, u_{i-1,\, j} \text{ , where } \lambda = \frac{k}{ah^2}.$$

It is valid only if $0 \le \lambda \le \frac{1}{2}$.

If $\lambda = \frac{1}{2}$ *Bender – Schmidt Formula is*

$$u_{i,\, j+1} = \frac{1}{2}\left[u_{i+1,\, j} + u_{i-1,\, j}\right] \quad \text{It is valid only if } k = \frac{a}{2}h^2$$

14. Write the Crank-Nicolson formula for solving Parabolic equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ (or) State Crank-Nicolson implicit scheme for solving the one dimensional heat flow equation using $k = ah^2$. (or) Express the simplest form of Crank-Nicolson scheme to solve $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$.

Solution: Implicit Formula is

$$\lambda\left(u_{i+1,j+1} + u_{i-1,j+1}\right) - 2\,(\lambda+1) u_{i,j+1} = 2(\lambda-1) u_{i,j} - \lambda\,(u_{i+1,j} + u_{i-1,j}), \quad \text{where } \lambda = \frac{k}{ah^2}$$

If $\lambda = 1$ *Crank – Nicholson Formula is*

$$u_{i,\, j+1} = \frac{1}{4}\left[u_{i-1,\, j+1} + u_{i-1,\, j} + u_{i+1,\, j+1} + u_{i+1,\, j}\right]$$

15. What is the value of K to solve $\frac{\partial u}{\partial t} = \frac{1}{2} u_{xx}$ by Bender Schmidt's with h = 1 if h and K are the increments of x and t respectively ?

Solution: Given $u_{xx} = 2u_t$ Here a = 2 , h = 1

$$\text{For Bender – Schimdt Wkt } \lambda = \frac{1}{2}$$

$$\therefore k = \frac{a}{2h^2} = \frac{2}{2(1)^2} = 1$$

16. What type of equations can be solved by using Bender Schmidt and Crank-Nicholson's difference formula ?

Solution: Bender Schmidt and Crank _ Nicholson's difference formula are used solve parabolic equations of the form $u_{xx} = au_t$.

17. Write different methods for solving boundary value problems.

Solution: i) Bender Schmidt method

ii) Crank- Nicholson method**.**

iii)Liebmann's Iteration Process.

.

18. How many initial and boundary conditions are required to solve the one dimensional wave equation.

Solution: Initial Conditions-2, Boundary Conditions-2.

19. Write down the explicit formula to solve the wave equation.

Solution: Explicit Formula is

$$u_{i,j+1} = 2(1-\lambda^2 a^2)u_{i,j} + \lambda^2 a^2 (u_{i-1,j} + u_{i+1,j}) - u_{i,j-1}$$

$$where\ \lambda = \frac{k}{h}$$

20. In the explicit formula for solving one dimensional wave equation, what is the simplest form to explicit scheme.

Solution: Explicit Formula is

$$u_{i,j+1} = 2(1-\lambda^2 a^2)u_{i,j} + \lambda^2 a^2 (u_{i-1,j} + u_{i+1,j}) - u_{i,j-1}$$

$$where\ \lambda = \frac{k}{h}$$

$$If\ \lambda = \frac{1}{a}, \quad u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

21. Write a note on the stability and convergence of the solution of the difference equation corresponding to the hyperbolic equation $u_{tt} = \alpha^2 u_{xx}$**.**

Solution: For $\lambda = \frac{1}{a}$, the solution of the difference equation is stable and coincides with the solution of the differential equation. For $\lambda > \frac{1}{a}$, the solution is unstable.

22. Write the Diagonal five point formula to solve the Laplace equation $u_{xx} + u_{yy} = 0$**.**

Solution: $u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1}\right]$

23. Write down the Standard five point formula to solve the Laplace equation (or) State Liebmann's iteration process formula.

Solution: $u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right]$

24. The number of conditions required to solve the Laplace equation is ________

Solution: ***Four.***

25. What is the purpose of Liebmann's process (or) principle?

Solution: The purpose of Liebmann's process is to find the solution of the Laplace equation $u_{xx}+u_{yy}=0$ by iteration over a square with boundary values and get correct values of the interior mesh points.

26. State Liebmann's iteration process formula.

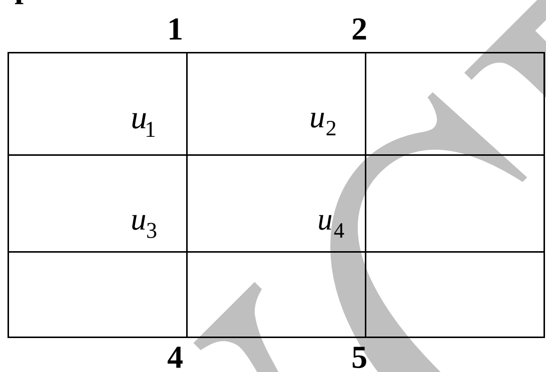
$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right]$$

27. Write down the difference scheme for the Poisson's equation $u_{xx}+u_{yy}=f(x,y)$.

Solution: We know that Poisson equation is $\nabla^2 u = f(x,y)$

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh)$$

28. For the following mesh in solving $\nabla^2 u = 0$ find one set of rough values of u at interior mesh points.



Solution: By symmetry , $u_2 = u_3$

Assume $u_2 = 0 \Rightarrow u_3 = 0$

$\therefore$ The rough values are $u_1 = \frac{1}{4}(1+1+u_2+u_3)$ **=0.5**

$$u_4 = \frac{1}{4}(u_2+u_3+5+5) = 2.5$$

$$u_3 = \frac{1}{4}(0.5+2+2.5+4) = 2.25 .$$

$$u_2 = \frac{1}{4}(0.5+2+2.5+4) = 2.25$$

$\therefore$ The Rough values are $u_1 = .5$, $u_2 = u_3 = 2.25$, $u_4 = 2.5$

PART - B

UNIT-I

SOLUTION OF EQUATIONS AND EIGENVALUE PROBLEMS

1. Solve by iteration method find the real root of $\cos x = 3x - 1.$
2. Using Newton-Rapshon method find the real root of

   $i) x\log_{10} x - 1.2 = 0.$ $ii) x\log_{10} x = 12.34$ start with $x_0 = 10$

   $iii) 2x - \log_{10} x - 7 = 0.$ $iv) 3x + \sin x = e^x$ $v) \cos x = xe^x$ $vi) x^3 - 6x + 4 = 0.$
3. Find the negative root of $x^3 - 2x + 5 = 0.$
4. Solve the following system by Gauss-Elimination method.

   $i)\ 2x + y + 4z = 12,\ 8x - 3y + 2z = 20,\ 4x + 11y - z = 33.$

   $ii)\ x - y + z = 1,\ -3x + 2y - 3z = -6,\ 2x - 5y + 4z = 5.$

   $iii)\ 5x - y = 9,\ -x + 5y - z = 4,\ -y + 5z = -6.$

   $iv)\ x + 2y - w = -2,\ 2x + 3y - z + 2w = 7,\ x + y + 3z - 2w = -6,\ x + y + z + w = 2.$
5. Solve the following system by Gauss-Jordan method.

   $i)\ 10x + y + z = 12,\ 2x + 10y + z = 13,\ x + y + z = 7.$

   $ii)\ x + 2y + z = 3,\ 2x + 3y + 3z = 10,\ 3x - y + 2z = 13.$

   $iii)\ 3x + 4y + 5z = 18,\ 2x - y + 8z = 15,\ 5x - 2y + 7z = 20.$

   $iv)\ 2x - y + 2z - w = -5,\ 3x + 2y + 3z + 4w = 7,\ x - 2y - 3z + 2w = 5,\ x + y + z + w = 2.$
6. Find the inverse of the matrix by Gauss-Jordan method.

$$i)\begin{pmatrix}1 & 1 & 3\\ 1 & 3 & -3\\ -2 & -4 & -4\end{pmatrix}\ ii)\begin{pmatrix}4 & 1 & 2\\ 2 & 3 & -1\\ 1 & -2 & 2\end{pmatrix}\ iii)\begin{pmatrix}2 & 0 & 1\\ 3 & 2 & 5\\ 1 & -1 & 0\end{pmatrix}\ iv)\begin{pmatrix}1 & 1 & 2\\ 1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}\ v)\begin{pmatrix}0 & 1 & 2\\ 1 & 2 & 3\\ 3 & 1 & 1\end{pmatrix}$$

7. Solve the following system by Gauss-Seidal method.

$i) 27x+6y-z=85$, $x+y+54z=110$, $6x+15y+2z=72.$

$ii) 20x+y-2z=17$, $3x+20y-z=-18$, $2x-3y+20z=25.$

$iii) 10x-5y-2z=3$, $x+6y+10z=-3$, $4x-10y+3z=-3.$

$iv) 8x-y+z=18$, $2x+5y-2z=3$, $x+y-3z=-6$

$v) 4x+2y+z=14$, $x+5y-z=10$, $x+y+8z=20.$

8. Find the dominant eigenvalues and the corresponding eigenvectors by Power Method.

$$i)\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \qquad ii)\begin{pmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{pmatrix}$$

9. Find all the eigenvalues by power method. $\begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

10. Using Gauss-Jacobi find the eigenvalues and eigenvectors of $i)\begin{pmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{pmatrix}$ $\quad ii)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$.

UNIT-II

INTERPOLATION AND APPROXIMATION

1. Using Lagrangian Interpolation Formula

i) Construct a polynomial for the following data f(0)= -12, f(1)= 0 , f(3)=6 & f(4)=12 Hence Evaluate f(2) , f(2.5) & f(3.5).

ii) Find f(x) from the following data

x	1	2	3	5
f(x)	0	7	26	124

iii) Find the polynomial & find f(4)

x	0	1	2	5
f(x)	2	3	12	147

iv) Find the value of f(4)

x	0	2	3	6
f(x)	-4	2	14	158

v) Find y at x = 6

x	3	7	9	10

f(x)	168	120	72	63

vi) Find y(2)

X	0	1	3	4	5
Y	0	1	81	256	625

vii) Find y(9.5)

x	7	8	9	10
Y	3	1	1	9

2. Newton's Divided Difference:

i) Find f(x) and find y when x=1

x	-1	0	2	3
y	-8	3	1	12

ii) Find f(x) and f(6)

x	1	2	7	8
f(x)	1	5	5	4

iii) Find the missing value from the table

X	1	2	4	5	6
Y	14	15	5	-	9

3. Newton's Forward Interpolation:

i) Find the Interpolating polynomial for y

x	4	6	8	10
y	1	3	8	16

ii) Using Suitable interpolation find f(1.5)

x	0	1	2	3	4
f(x)	858.3	869.6	880.9	892.3	903.6

iii) Find $e^{-1.1}$.

x	1.00	1.25	1.50	1.75	2.00
e^{-x}	0.3679	0.2865	0.2231	0.1738	0.1353

iv) Find the number of students between the weights 60 and 70

Weight x	0-40	40-60	60-80	80-100	100-120
No.of Students	250	120	100	70	50

v) Find $U_{1/2}$ given $U_{-1} = 202, U_0 = 175, U_1 = 82, U_2 = 55.$

4. Newton's Backward Interpolation:

Find the value of y when x=27.

x	10	15	20	25	30
y	35.4	32.2	29.1	26	23.1

5. Newton's Forward and Backward Formula:

i)Find the melting point of alloy contains lead when x=42, x=43 x=48 and x=84.

x	40	50	60	70	80	90
θ	184	204	226	250	276	304

ii) The population of a town in census is as given below. Estimate the population in 1996 using Newton's Forward and Backward Interpolation.

Year	1961	1971	1981	1991	2001
Population in 1000's	46	66	81	93	101

iii)The following are taken from steam table. Find the pressure at temperature $t=142^0C$ and $t=175^0C$.

Temp ^{0} C	140	150	160	170	180
P kg/cm^2	3.685	4.854	6.302	8.076	10.225

iv) From the table of half-yearly premium for policies mature at different ages 46 and 63.

Age x	45	50	55	60	65
Premiun	114.84	96.16	83.32	74.48	68.48

6. Cubic Spline:

i) Obtain the cubic spline approximation for the function y=f(x) from the following table, given that $y_0'' = y_2'' = 0$.

x	-1	0	1	2
y	-1	1	3	35

ii) Using cubic spline find y(0.5) given $M_0=M_2=0$ and the table

x	0	1	2
y	-5	-4	3

iii) Find a natural cubic spline to the following data. Compute y(1.5) y(1.75) and y'(1).

x	1	2	3
y	-8	-1	18

iv) Given the points $(0,0),\left(\frac{\pi}{2},1\right)$ & $(\pi,0)$ satisfying the function y=sinx $(0\le x\le\pi)$ determine the value of $y\left(\frac{\pi}{6}\right)$ using cubic spline approximation.

UNIT – III

NUMERICAL DIFFERENTIATION AND INTEGRATION

1.Derivatives:

i). Find the value of sec 31^0 using the following.

x^0	31	32	33	34
$\tan x^0$	0.6008	0.6249	0.6494	0.6748

ii)The population of a certain town is given below. Find the rate of growth of the population in 1931,1971.

Year x:	1931	1941	1951	1961	1971
Population in 1000's y:	40.62	60.80	79.95	103.56	132.65

iii) Find the first, second and third derivatives of the function at the point x=1.5 and x=4.0.

x	1.5	2.0	2.5	3.0	3.5	4.0
y	3.375	7	13.625	24	38.875	59

iv) Find $f'(3)$ *and* $f''(3)$ from the following:

x	3	3.2	3.4	3.6	3.8	4.0
y	-14	-10.032	-5.296	-0.256	6.672	14

v) The following data gives the velocity of a particle for 20 seconds at an interval of 5 seconds. Find the initial acceleration using the entire data.

Time (sec):	0	5	10	15	20
Velocity (m/sec)	0	3	14	69	228

vi) Find the value of f'(8) from the following:

x	6	7	9	12
y	1.556	1.69	1.608	2.158

vii) Find y' at x=51 from

x	50	60	70	80	90
y	19.96	36.65	58.81	77.21	94.61

2. Trapezoidal Rule, Simpson's 1/3 rule and Simpson's 3/8 rule:

i) Evaluate $\int_0^\pi \sin x dx$ by dividing the range into ten and six equal parts using Trapezoidal and Simpson's Rule.

ii) Evaluate $\int_0^6 \frac{1}{1+x^2} dx$ by using Trapezoidal and Simpson's Rule.

iii) Evaluate $\int_0^1 \frac{1}{1+x^2} dx$ by using Trapezoidal and Simpson's Rule.

iv) Evaluate $\int_0^1 \frac{\sin x}{x} dx$ by dividing the range into 6 equal parts using Simpson's 1/3 Rule.

v) Evaluate $\int_0^5 \frac{1}{4x+5} dx$ by Simpson's 1/3 rule and hence find the value of $\log_e 5$. (n=10).

vi) Evaluate $\int_0^6 \frac{1}{1+x} dx$ by using Trapezoidal and Simpson's Rule check it by direct integration.

3. Romberg's Method:

i) Evaluate $\int_0^2 \frac{1}{4+x^2} dx$ using Romberg's method. Hence find the approximate value of π.

ii)Evaluate I= $\int_0^1 \frac{1}{1+x^2} dx$ correct to 4 decimal places by using Romberg's method. Hence find the approximate value of π.

4. Double Integral:

i) Evaluate $\int_0^1 \int_0^1 \frac{1}{x+y+1} dxdy$ by Trapezoidal rule taking h=0.5, k=0.25.

ii) Evaluate $\int_4^{4.4} \int_2^{2.6} \frac{1}{xy} dydx$ by Trapezoidal rule.

iii) Evaluate $\int_0^{0.5} \int_0^1 e^{xy} dxdy$ by Simpson's rule taking h=0.5, k=0.25.

iv) Evaluate $\int_1^2 \int_1^2 \frac{xy}{x+y} dxdy$ by Trapezoidal rule taking h= k=0.25.

5. Gaussian Quadrature:

i) Evaluate $\int_0^1 \frac{1}{1+x^2} dx$ by 2 point and 3 point Gaussian Quadrature formula and hence find the approximate value of π.

ii) Evaluate I= $\int_{-1}^1 \frac{1}{1+x^2} dx$ by 2 point and 3 point Gaussian Quadrature formula

iii) Evaluate $\int_5^{12} \frac{1}{x} dx$ by 2 point and 3 point Gaussian Quadrature formula.

iv) Evaluate $\int_{0.2}^{1.5} e^{-x^2} dx$ by 2 point and 3 point Gaussian Quadrature formula.

UNIT-IV

INITIAL VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

1. Solve $\frac{dy}{dx} = x^2 + y^2,\ y(0) = 1.$ Use Taylor series method at x=0.2 and x=0.4.

2. Solve $y' - x^2 y + 1 = 0,\ y(0) = 1.$ Find y(0.2) and y(0.4) by Taylor series method.

3. Using Taylor's method compute y(0.2) and y(0.4) correct to 4 decimal places given $\frac{dy}{dx} = 1 - 2xy,\ y(0) = 0.$

4. By Taylor's Method find y(0.1) given that $y'' = y + xy',\ y(0) = 1,\ y'(0) = 0.$

5. Solve $\frac{dy}{dx} = x^2 - y,\ y(0) = 1.$ by Modified Euler's method for x=0.2,x=0.4 in steps of 0.2 each.

6. Using Euler's method find y(0.2) and y(0.4) from y'=x+y, y(0)=1 with h=0.2.

7. Find y(0.2) ,y(0.4). Given y'=y+e^x, y(0)=0 by Modified Euler's Method.

8. Compute y(0.2) and y(0.4) from $y' = \frac{y^2 - x^2}{y^2 + x^2}, y(0) = 1$ by fourth order Runge-kutta method taking h=0.2.

9. Given $\frac{dy}{dx} = \frac{y^2 - 2x}{y^2 + x}$ and y=1 when x=0. Find y(0.2) by using 4th order R-K method with h=0.2

10. Solve $\frac{dy}{dx} = x^2 + y,\ y(0) = 2.$ Compute y(0.2),y(0.4) and y(0.6) by 4th order R-K Method.

11. Using 4th order R-K Method to determine y(0.2) with h=0.1 from $\frac{dy}{dx} = x^2 + y^2,\ y(0) = 1.$

12. Given 5xy'+y^2=2, y(4)=1, y(4.1)=1.0049, y(4.2)=1.0097, y(4.3)=1.0143. Compute y(4.4) using Milne's Method.

13. Given first order ordinary differential equation y'=xy+y^2, y(0)=1, y(0.1)=1.1169, y(0.2)=1.2773.

iv) Find y(0.3) by R-K 4th order method.

v) Find y(0.4) by Milne's Method.

14. Determine the value of y(0.4) using Milne's method given y'=xy+y^2, y(0)=1. Use Taylor's series to get the values of y(0.1), y(0.2) and y(0.3).

15. Given initial values problem $\frac{dy}{dx} = \frac{1}{2}(x + y),\ y(0) = 2.$ y(0.5)=2.636, y(1)=3.595, y(1.5)=4.968. Find y(2) by Adam's Method.

16. $\frac{dy}{dx} = xy + y^2,\ y(0)=1, y(0.1)=1.1169, y(0.2)=1.2774, y(0.3)=1.5041.$ Find y(0.4) using Adam's method.

UNIT-V

BOUNDARY VALUE PROBLEMS IN ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

1) Solve $y'' - xy = 0,\ y(0) = -1,\ y(1) = 2$ by finite difference method taking h=1/3 and n=2.

2) Solve $xy'' + y = 0,\ y(1) = 1,\ y(2) = 2$ with h=0.5 by finite difference method.

3) Solve $\frac{d^2y}{dx^2} + xy = 1,\ y(0) = 0,\ y'(1) = 1$ with n=2 (take h=0.5) by finite difference method.

4) Solve $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial t} = 0$. Given u(0,t)=0, u(4,t)=0, u(x,0)=x(4-x) and assuming h=1. Find the values of u upto t=5 seconds by Schmidt method.

5) Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with the conditions u(0,t)=u(1,t)=0 for t>0 and u(x,0)=x(1-x), 0<x<1 using explicit method, with $\Delta x = 0.2$ for 5 time steps.

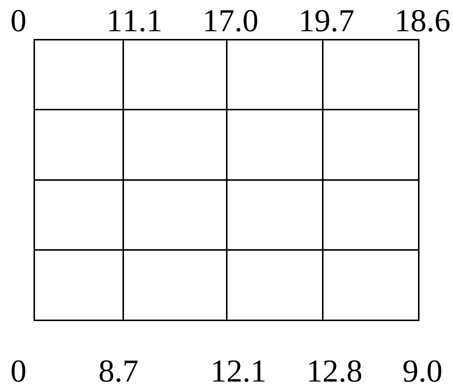
6) Solve $u_{xx} = 16u_t,\ 0 < x < 1, t > 0$ given u(x,0)=0=u(0,t) and u(1,t)=100t. Choose h=1/4 by Crank-Nicholson method.

7) Solve the equation $u_t = u_{xx},\ 0 \le x \le 1, t > 0$ by Crank-Nicholson method under the condition u(0,t)=u(1,t)=0 and $u(x,0) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2(1-x), & \frac{1}{2} \le x \le 1 \end{cases}$ for one time step.

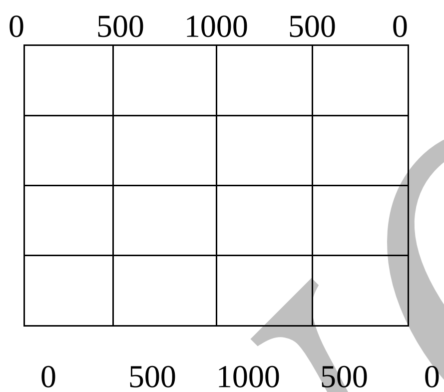
8) Solve $4u_{xx} = u_{tt}$ with boundary conditions u(0,t)=0, u(4,t)=0, $u_t(x,0)=0$ and u(x,0)=x(4-x). Taking h=1, t=4 seconds.

9) Solve $25u_{xx} = u_{tt}$ for u at the pivotal points given u(0,t)=0, u(5,t)=0, $u_t(x,0)=0$ and $u(x,0) = \begin{cases} 2x, & 0 \le x \le 2.5 \\ 10-2x, & 2.5 \le x \le 5 \end{cases}$ for one half period of viberation.

10) Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ at the nine mesh points of the square given below. The values of u at the boundaries are specified in the figure.



11) Solve $\nabla^2 u = 0$ for the following mesh with boundary values by Liebmann's scheme.



12) Solve $u_{xx} + u_{yy} = 0$ over the square mesh of side 4 units satisfying the following conditions.

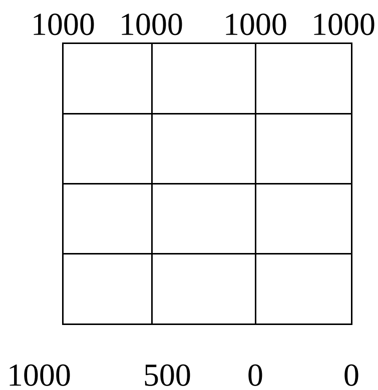
$$i) u(0,y) = 0, \quad for\ 0 \le y \le 4$$
$$ii) u(4,y) = 12 + y, \quad for\ 0 \le y \le 4$$
$$iii) u(x,0) = 3x, \quad for\ 0 \le x \le 4$$
$$iv) u(x,4) = x^2, \quad for\ 0 \le y \le 4$$

13) Solve $u_{xx} + u_{yy} = 0,\ 0 \le x \le 4,\ 0 \le y \le 4,\ u(0,y) = 0,\ u(4,y) = 16,$ $u(x,0) = 0,\ u(x,4) = x^3.$ Divide the square plate into 16 square meshes of side h=1. Obtain the result correct to one places of decimal.

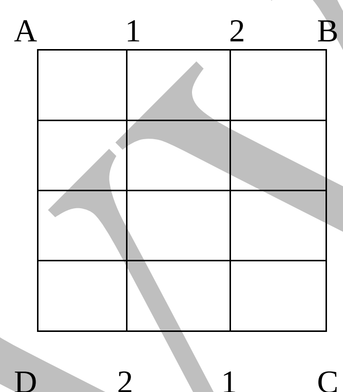
14) Solve $\nabla^2 u = 0$ in the square region bounded by x=0,x=4,y=0,y=4 and with boundary conditions $u(0,y) = \frac{y^2}{2},\ u(4,y) = y^2,\ u(x,0) = 0,\ u(x,4) = 8 + 2x.$ taking h=k=1.

15) Solve $u_{xx} + u_{yy} = 0, \quad 0 \le x \le 1,$ with u(0,y)=10=u(1,y), u(x,0)=20=u(x,1). Take h=0.25 to 3 decimal accuracy.

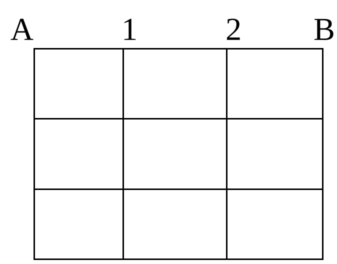
16) Evaluate the function u(x,y) satisfy $\nabla^2 u = 0$ at the lattice points given the boundary values as follows.

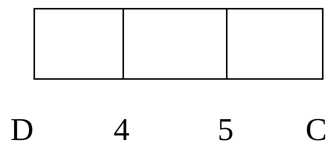


17) Solve $u_{xx} + u_{yy} = 0$ for the following square with boundary conditions as shown below,



18) Solve $u_{xx} + u_{yy} = 0$ for the following square with boundary conditions as shown below,





19) Solve $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square mesh with sides x=0,x=3,y=0,y=3 with u=0 on the boundary and mesh length is 1 unit.